

Functional Approach to Classical Yang-Mills Theories

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Sometime ago it was shown that the operatorial approach to classical mechanics, pioneered in the 30's by Koopman and von Neumann, can have a functional version. In this talk we will extend this functional approach to the case of classical field theories and in particular to the Yang-Mills ones. We shall show that the issues of gauge-fixing and Faddeev-Popov determinant arise also in this classical formalism.

1. Introduction

In this talk we want to apply the formalism of the Classical Path Integral (CPI) [1] to a classical field theory with non-abelian gauge invariance. The reasons to embark on this work are basically two. The first is geometrical. As it has been explained in a paper by E. Gozzi in these proceedings [2], the CPI provides several geometrical tools like forms, exterior derivatives, Lie-derivatives etc. These tools may help us in probing the geometrical aspects of Yang-Mills (YM) theories which play an important role in the study of anomalies and similar phenomena. The second reason to spend time on the classical aspects of field theories is because the classical solutions are the basic ingredients of non-perturbative approaches to quantum field theory which are needed to study phenomena like confinement, mass spectrum calculations and so on. The gauge invariance of YM theories creates several problems in the implementation of the CPI procedure; so we prefer to start with the simple example of a scalar field φ with a φ^4 interaction:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2 \varphi^2 - \frac{1}{24}g\varphi^4 \quad (1)$$

Calling π the momentum conjugate to φ we can easily perform the Legendre transform and construct the associated Hamiltonian:

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_r \varphi)^2 + \frac{1}{2}m^2 \varphi^2 + \frac{1}{24}g\varphi^4 \right] \quad (2)$$

The equations of motion can be written in the following form: $\dot{\varphi}(x) = \frac{\delta H}{\delta \pi(x)}$, $\dot{\pi}(x) = -\frac{\delta H}{\delta \varphi(x)}$. We note a first formal difference between the classical mechanics of a point particle and the classical field theory: the presence of functional derivatives in the equations of motion. Calling the field and its conjugate momentum as $\xi^a = (\varphi, \pi)$, with $a = 1, 2$ and introducing a 2×2 antisymmetric matrix ω^{ab} we can write the equations of motion in the following compact form: $\dot{\xi}^a = \omega^{ab} \frac{\delta H}{\delta \xi^b}$. We can then go along the usual steps of the CPI procedure [2] in order to have an exponential weight in the path integral:

$$Z_{CPI}^{\varphi^4} = \int \mathcal{D}\xi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a e^{i \int d^4x \tilde{\mathcal{L}}^{\varphi^4}} \quad (3)$$

where the *Lagrangian density* has the same form as that associated to the CPI of the point particle:

$$\begin{aligned} \tilde{\mathcal{L}}^{\varphi^4} = & \lambda_a (\dot{\xi}^a - \omega^{ab} \frac{\delta H}{\delta \xi^b}) \\ & + i\bar{c}_a (\partial_t \delta_b^a - \omega^{ac} \frac{\delta}{\delta \xi^c} \frac{\delta}{\delta \xi^b} H) c^b \end{aligned} \quad (4)$$

What are then the main features that make the CPI of a point particle different from the CPI of a field theory? The fact that a field has ∞ degrees of freedom (one for every point of the space \vec{x}) has several consequences: **a)** we have to label the fields with an \vec{x} ; **b)** the Lagrangian density $\tilde{\mathcal{L}}$ in (3) is integrated over d^4x and not

only over the time variable; **c**) in the functional measure $\mathcal{D}\xi^a = \prod_{k, \vec{x}} d\xi^a(k, \vec{x})$ the product is extended also to space variables.

Having shown the basic formal aspects of a simple CPI field theory, we can now go on with the case of Yang-Mills theories. The main ingredients are: **1**) the fields $A_a^\mu(x)$ with a colour index running from 1 to n , being n the number of generators T_a of a Lie algebra; **2**) the structure constants C_a^{bc} appearing in the algebra of the T_a : $[T_a, T_b] = iC_{ab}^c T_c$; **3**) the antisymmetric tensor $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gC_a^{bc} A_b^\mu A_c^\nu$ which enters into the Lagrangian density in the usual way: $\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a$.

2. A natural gauge-fixing procedure

If we want to construct, as in the case of a φ^4 theory, the associated Hamiltonian we have to solve the problem of the constraints. The *primary* ones derive from the definition itself of conjugate momenta: $\pi_a^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_a^0} = 0$. The *secondary* ones $\sigma_a = -\vec{\nabla} \cdot \vec{\pi}_a + C_{ac}^b \pi_b^k A_k^c = 0$ derive instead from the requirement that the primary constraints have to be conserved under time evolution. Because of these constraints we cannot write the Hamiltonian in a *unique* way. In fact, following [3], we can write the Hamiltonian as:

$$H = \int d^3x \left(\frac{1}{2} \vec{\pi}_a \cdot \vec{\pi}_a + \frac{1}{2} \vec{B}_a \cdot \vec{B}_a + \lambda^a \sigma_a \right) \quad (5)$$

It is given by the sum of the YM Hamiltonian $\mathcal{H}_{YM} = \frac{1}{2} \vec{\pi}_a \cdot \vec{\pi}_a + \frac{1}{2} \vec{B}_a \cdot \vec{B}_a$ and an arbitrary combination of the secondary constraints $\lambda^a \sigma_a$ where $B_{ai} = \frac{1}{2} \epsilon_{ijk} F_a^{jk}$ and λ^a are n Lagrangian multipliers identical to the first component of the gauge potentials $\lambda^a = A_0^a$.

In the 50's Dirac [4] recognized the necessity of a gauge-fixing procedure, already at the classical level, when he tried to give an hamiltonian formulation for a system with first class constraints. This necessity is particularly evident in the CPI approach. In fact from the Hamiltonian (5) we can derive the following equations of motion:

$$\begin{aligned} \dot{\pi}_a^k &= -\partial_i F_a^{ki} - \lambda^d C_{da}^b \pi_b^k + C_{ac}^b A_i^c F_b^{ki} \\ \dot{A}_k^a &= \pi_k^a + \partial_k \lambda^a + C_{dc}^a \lambda^d A_k^c \end{aligned} \quad (6)$$

Let us indicate with: $\phi^A = (A_k^a, \pi_a^k)$ both the spatial components of the fields and the momenta. We can then rewrite the equations of motion in terms of a $6n \times 6n$ antisymmetric matrix ω^{AB} : $\dot{\phi}^A = \omega^{AB} \frac{\partial H}{\partial \phi^B}$ where we indicate with ∂ the functional derivative and H is given by eq. (5). At this point in the CPI procedure the step $\tilde{\delta}(\phi^A - \phi_{cl}^A) \rightarrow \tilde{\delta}(\dot{\phi}^A - \omega^{AB} \frac{\partial H}{\partial \phi^B})$ is *forbidden* since, because of the arbitrariness of λ^a , there is not a one-to-one correspondence between solutions and equations of motion. So we have to *fix the gauge*. We can, for example, enforce the Lorentz-gauge condition by inserting into the original path integral:

$$Z_{CPI}^{YM} = \int \mathcal{D}\phi \tilde{\delta}(\phi^A - \phi_{cl}^A) \quad (7)$$

the following expression formally independent of the fields:

$$1 = \int \mathcal{D}A_0^a \tilde{\delta}(-\partial^\mu A_\mu^a) \Delta_F[\mathbf{A}] \quad (8)$$

where $\Delta_F[\mathbf{A}] \sim \det[-\partial^\mu \delta_c^a D_\mu^{bc}]$ is the usual Faddeev-Popov determinant. Having fixed the gauge, we can now perform the transformation from the solutions to the equations of motion in (7):

$$\begin{aligned} \tilde{\delta}(\phi^A - \phi_{cl}^A) &= \tilde{\delta}(\dot{\phi}^A - \omega^{AB} \partial_B H) \\ &\quad \cdot \det(\delta_B^A \partial_t - \omega^{AC} \partial_B \partial_C H) \end{aligned} \quad (9)$$

where all the derivatives have to be intended as functional ones. Next we have to exponentiate not only the RHS of (9) but also the gauge-fixing part (8) in order to give an exponential weight to the CPI. We will name this CPI version of YM theories as NAT (natural) because this is the most natural and direct way of fixing the gauge:

$$Z_{CPI}^{NAT} = \int \mathcal{D}\mu \exp i \int d^4x \tilde{\mathcal{L}}^{NAT} \quad (10)$$

The functional integration is extended over all the variables μ of the theory (see also the next section) and the Lagrangian density is:

$$\begin{aligned} \tilde{\mathcal{L}}^{NAT} &= -\pi_a \partial^\mu A_\mu^a + \Lambda_A (\dot{\phi}^A - \omega^{AB} \partial_B H) \\ &\quad - i \partial^\mu \bar{C}_a D_{b\mu}^a C^b + i \bar{\Gamma}_A (\delta_B^A \partial_t - \omega^{AC} \partial_B \partial_C H) \Gamma^B \end{aligned} \quad (11)$$

In (11) π_a is the variable we use to exponentiate the gauge fixing condition, \bar{C}_a and C^b are the usual Faddeev-Popov ghosts, while the $(\Lambda, \Gamma, \bar{\Gamma})$ are the analogue of the (λ, c, \bar{c}) of the CPI of the point particle [1].

3. BFV method

In quantum field theory there is also a more general way of implementing the gauge-fixing procedure: the Batalin, Fradkin and Vilkovisky (BFV) method. In this approach one enlarges the original phase-space (A_k^a, π_a^k) to include as dynamical variables also the Lagrangian multipliers λ^a , their conjugate momenta π_a , and a number of BFV ghosts $\eta^a = (-iP^b, C^b)$ and antighosts $\mathcal{P}_a = (i\bar{C}_b, \bar{P}_b)$ equal to the number of constraints $\psi_a = (\pi_b, \sigma_b)$ present in the theory. All the ghosts and the constraints contribute in building the following BRS-BFV charge:

$$\Omega^{BFV} = \int d^3x [\sigma_a C^a - iP^a \pi_a + \frac{1}{2} \bar{P}_a C_{bc}^a C^b C^c] \quad (12)$$

For reasons that are clearly explained in [3]-[5], we have to identify two Hamiltonian densities that differ for a BRS-exact term: $\mathcal{H}^{BFV} = \mathcal{H}_{YM} - \{\theta, \Omega^{BFV}\}$, where the $\{\cdot, \cdot\}$ are the graded Poisson brackets that one can introduce in the BFV phase-space. In this case fixing the gauge means choosing a gauge function θ . If we choose: $\theta = i\bar{C}_a \partial^k A_k^a + \bar{P}_a \lambda^a$ then we obtain the following Hamiltonian density:

$$\begin{aligned} \mathcal{H}^{BFV} = & \frac{1}{2} \pi_a^k \pi_k^a + \frac{1}{4} F_a^{ij} F_{ij}^a + \pi_a \partial^k A_k^a \\ & - \lambda^a \partial_k \pi_a^k + \lambda^a C_{ac}^b \pi_b^k A_k^c + i\bar{P}_a P^a \\ & - \lambda^a \bar{P}_b C_{ac}^b C^c - i\bar{C}_a \partial^k (\partial_k C^a + C_{bc}^a A_k^b C^c) \end{aligned} \quad (13)$$

As the BFV method has automatically managed to implement the gauge-fixing procedure, we can then proceed to build the CPI without adding any further ingredient. First, we can derive the equations of motion and put them in the usual symplectic form: $\dot{\xi}^A = \omega^{AB} \bar{\partial}_B H^{BFV}$. We indicate with ξ^A all the fields, including the gauge ghosts, present in $H^{BFV} = \int d^3x \mathcal{H}^{BFV}$. So the grassmannian character of the fields is mixed: there are grassmannian even as well as odd fields. This

will cause some formal complications. For example to pass from the solutions to the equations of motion we need an object that is known in literature as the superdeterminant or berezinian. The superdeterminant of a general supermatrix M_B^A can be exponentiated [6] using auxiliary variables which have a grassmannian parity opposite to that of the fields they refer to:

$$sdet(M_B^A) = \int \mathcal{D}\bar{\Gamma} \mathcal{D}\Gamma \exp - \int d^4x \bar{\Gamma}_A M_B^A \Gamma^B \quad (14)$$

where $[\Gamma^A] \equiv [\Gamma^{\xi^A}] = [\bar{\Gamma}_{\xi^A}] = [\xi^A] + 1$ (we indicate in square brackets the grassmannian parity). In the same way the exponentiation of the equations of motion can be done using suitable auxiliary variables Λ_A , which have the same grassmannian parity with respect to the fields they refer to: $[\Lambda_A] \equiv [\Lambda_{\xi^A}] = [\xi^A]$. In this way we can construct the following CPI for YM theory [6]:

$$Z_{CPI}^{BFV} = \int \mathcal{D}\Lambda_A \mathcal{D}\xi^A \mathcal{D}\bar{\Gamma}_A \mathcal{D}\Gamma^A e^{i \int d^4x \tilde{\mathcal{L}}^{BFV}} \quad (15)$$

where:

$$\tilde{\mathcal{L}}^{BFV} = \Lambda_A \dot{\xi}^A + i\bar{\Gamma}_A \dot{\Gamma}^A - \tilde{\mathcal{H}}^{BFV} \quad (16)$$

and:

$$\tilde{\mathcal{H}}^{BFV} = \Lambda_A \omega^{AB} \bar{\partial}_B H + i\bar{\Gamma}_A \omega^{AC} \bar{\partial}_C H - \bar{\partial}_B \Gamma^B \quad (17)$$

Somehow we have constructed *two* different classical path integrals for Yang-Mills theories: the natural one, Z_{CPI}^{NAT} of eq. (10), and the BFV one, Z_{CPI}^{BFV} of eq. (15). The functional integrations entering the Z_{CPI}^{NAT} and Z_{CPI}^{BFV} are over a different number of fields. We have proved in [7] that several integrations in Z_{CPI}^{BFV} can be done explicitly to bring: $Z_{CPI}^{BFV} = \int \mathcal{D}\mu' \exp i \int d^4x \tilde{\mathcal{L}}^{BFV}$ down to: $Z_{CPI}^{NAT} = \int \mathcal{D}\mu \exp i \int d^4x \tilde{\mathcal{L}}^{NAT}$. In a certain sense we can say that this is the classical counterpart of the proof of the equivalence of the Faddeev-Popov and BFV methods in quantum field theory [3].

4. CPI superfields in YM theories

In the following table we will write down all the fields of the CPI-BFV theory (15); among all

these fields we shall indicate with bold characters those ones which do not enter the Z_{CPI}^{NAT} :

$$\begin{array}{cccccccc} A_k^a & \pi_a^k & \lambda^a & \pi_a & C^a & \bar{C}_a & \mathbf{P}_a & \bar{\mathbf{P}}_a \\ \Gamma_k^{A^a} & \Gamma_{\pi_a^k} & \mathbf{\Gamma}_{\lambda^a} & \mathbf{\Gamma}_{\pi_a} & \mathbf{\Gamma}_{C^a} & \mathbf{\Gamma}_{\bar{C}_a} & \mathbf{\Gamma}_{P_a} & \mathbf{\Gamma}_{\bar{P}_a} \\ \bar{\Gamma}_k^{A^a} & \bar{\Gamma}_{\pi_a^k} & \bar{\mathbf{\Gamma}}_{\lambda^a} & \bar{\mathbf{\Gamma}}_{\pi_a} & \bar{\mathbf{\Gamma}}_{C^a} & \bar{\mathbf{\Gamma}}_{\bar{C}_a} & \bar{\mathbf{\Gamma}}_{P_a} & \bar{\mathbf{\Gamma}}_{\bar{P}_a} \\ \bar{\Lambda}_k^{A^a} & \bar{\Lambda}_{\pi_a^k} & \bar{\Lambda}_{\lambda^a} & \bar{\Lambda}_{\pi_a} & \bar{\Lambda}_{C^a} & \bar{\Lambda}_{\bar{C}_a} & \bar{\Lambda}_{P_a} & \bar{\Lambda}_{\bar{P}_a} \end{array}$$

The fields that appear in the same vertical line in the previous table can be put together and considered as the components of a multiplet which was called the CPI superfield in ref. [1]. With respect to the point particle case we need to introduce two different superfields: an *even* superfield defined as:

$$\Xi^a = \xi^a + \theta \Gamma^a + \bar{\theta} \omega^{ab} \bar{\Gamma}_b + i \bar{\theta} \theta \omega^{ab} \Lambda_b \quad (18)$$

and an *odd* one defined as:

$$\Xi^\alpha = \xi^\alpha - \theta \Gamma^\alpha - \bar{\theta} \omega^{\alpha\beta} \bar{\Gamma}_\beta - i \bar{\theta} \theta \omega^{\alpha\beta} \Lambda_\beta \quad (19)$$

Notice that they differ for some signs. We can then easily show that the \mathcal{H}^{BFV} and the associated $\tilde{\mathcal{H}}^{BFV}$ are related as follows:

$$i \int d\theta d\bar{\theta} \mathcal{H}^{BFV}(\Xi) = \tilde{\mathcal{H}}^{BFV} \quad (20)$$

The concept of superfield allows us also to easily map the quantum symmetry charges into the corresponding ones of the CPI. For example, from Ω^{BFV} we can construct the correspondent $\tilde{\Omega}^{BFV}$ charge as:

$$i \int d\theta d\bar{\theta} \Omega^{BFV}(\Xi) = \tilde{\Omega}^{BFV} \quad (21)$$

It is possible to prove that the conservation of Ω^{BFV} under \mathcal{H}^{BFV} implies the conservation of $\tilde{\Omega}^{BFV}$ under the evolution generated by $\tilde{\mathcal{H}}^{BFV}$. Similarly to what happens in the quantum case, the classical action $\tilde{S}^{BFV} = \int d^4x \tilde{\mathcal{L}}^{BFV}$ is invariant under the transformations generated by $\tilde{\Omega}^{BFV}$. This property is crucial to *guarantee the gauge-invariance* of the physical results at the classical level.

The *universal* charges of the CPI found for the point particle can be found also in the BFV version of the YM-CPI. For example the BRS charge of the CPI becomes:

$$Q_{CPI}^{BRS} = i \int d^3x \Gamma^A \Lambda_A \quad (22)$$

The physical meaning of Q_{CPI}^{BRS} and $\tilde{\Omega}^{BFV}$ is very different. In fact, if we return to the table of all the fields of the BFV method, we have that the action of $\tilde{\Omega}^{BFV}$ on the fields of the first row is identical to that of the quantum charge Ω^{BFV} . They both generate the gauge transformations of the theory and mix the fields horizontally. Anyhow, differently than Ω^{BFV} , the $\tilde{\Omega}^{BFV}$ acts also on the fields of the other rows mixing them. The Q_{CPI}^{BRS} charge, instead, turns the fields of the first row into the associated Jacobi fields contained in the second row and so it allows us to move vertically through the table. Even if we called both charges as BRS charges they actually perform different operations.

The reader may ask which is the goal of all the machinery we have built. What we have in mind is a geometrical goal. In fact in ref. [8] it was proved that, choosing proper boundary conditions, the CPI of the point particle can be turned into a topological field theory and used to calculate, among other things, the Euler number of the manifold on which the system lives. Performing the same thing with the Z_{CPI}^{YM} we could in principle calculate the analog of the Euler number for the space of gauge orbits or other geometrical characteristics of this space. All these geometrical features are crucial for a better grasp of the non perturbative regime of the YM theories.

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